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1988 J. Phys. A: Math. Gen. 21 1477

(http://iopscience.iop.org/0305-4470/21/6/024)

J. Phys. A: Math. Gen. 21 (1988) 1477-1482. Printed in the UK

COMMENT

Asymptotic form of the spectral dimension at the fractal to lattice crossover

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Received 7 December 1987

Abstract. We study the spectral dimension of the Sierpinski gasket family of fractals. Each member of the family is labelled by an integer b ($2 \leqslant b < \infty$), so that when $b \to \infty$ both the fractal and spectral dimension approach the Euclidean value 2. The asymptotic law for the spectral dimension was recently suggested to be $d_s \simeq 2 - B/(\ln b)^{\beta}$, with B and β being some constants. Here we demonstrate that this form should be replaced by the asymptotic law $d_s \simeq 2 - \ln(\ln b) + \text{constant}/\ln b$. Our analysis is based on the exact calculation of the electric resistances R_b for all members of the family up to $b = 650$.

What governs the dynamical properties of a fractal lattice when it is almost Euclidean? This interesting problem has been attacked by Borjan *et al* (1987), who studied the spectral dimension d_s of a family of the Sierpinski type of fractals (Hilfer and Blumen 1984). The members of the family are labelled and characterised by an integer b, $2 \leqslant b < \infty$, so that $b = 2$ corresponds to the Sierpinski gasket and $b = \infty$ to a wedge of the ordinary triangular lattice. When $b \to \infty$, the fractal dimension of the family approaches the Euclidean dimension $d = 2$ via the asymptotic law

$$d_f \simeq 2 - \ln 2/\ln b. \tag{1}$$

By studying electrical resistances of the fractals, Borjan *et al* (1987) calculated exact values of d_s for $2 \leqslant b \leqslant 200$. They proposed the asymptotic law

$$d_s \simeq 2 - \frac{B}{(\ln b)^{\beta}} \tag{2a}$$

where B is a constant. Here we argue that this is not the correct form but must be replaced by

$$d_s \simeq 2 - \frac{\ln(\ln b)}{\ln b} + \frac{\text{constant}}{\ln b}. \tag{2b}$$

We obtain (2b) by calculating the electrical resistances of a much longer sequence ($2 \leqslant b \leqslant 650$) for the members of the fractal family.

In order to determine exact values of the spectral dimension d_s, for various b, we use the general relation

$$d_s \equiv \frac{2d_f}{d_w} = \frac{2d_f}{(d_f + \tilde{\zeta})} \tag{3}$$

where $\tilde{\zeta}^*$is the scaling exponent of the DC resistance, which is in the case under study determined by (Borjan *et al* 1987)

$$\tilde{\zeta}=\frac{\ln(\frac{3}{2}R_b)}{\ln b} \tag{4}$$

with R_b being the DC resistance of the fractal generator. The latter is an equilateral triangle that contains b^2 identical smaller triangles whose sides are unit resistors (see figure 1(a)). Hence, the overall behaviour of d_s, including the large b limit, is determined by R_b. We have calculated particular values of R_b by applying the Δ-Y and Y-Δ transformations (Lobb and Frank 1984, Borjan *et al* 1987), by which each fractal generator can be, in principle, reduced to a single resistor. In practice, by writing a convenient computer algorithm for the fractal generator reduction, we have been able to calculate, on the IBM 3090 mainframe at Boston University, R_b for all b up to $b=650$. A subsequence of the obtained data is listed in table 1. The main limitation for getting bigger values for b is imposed by the limited available space of the computer. In fact, in order to expand our range of values up to 650 we had to rewrite the program paying attention to the memory efficiency. The new algorithm was almost twice as efficient.

Since it is clear that the large b limit of R_b determines the asymptotic behaviour of the exponent $\tilde{\zeta}$, and consequently of the spectral dimension d_s, we shall first discuss

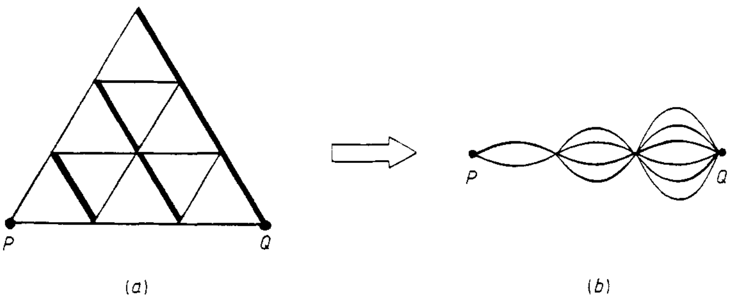

Figure 1. (a) The $b=3$ fractal generator. An electric DC current is sent into vertex P and taken out of the vertex Q. It is assumed that each bond (18 altogether) of each unit triangle is a resistor of a unit resistance. The equivalent resistance of the entire network can be calculated by the Δ-Y and Y-Δ transformations (see, for instance, Borjan *et al* 1987) and is equal to $\frac{30}{21}$ (in arbitrary units). However, when the zero resistance bonds are substituted for all unit resistance bonds that are leaned to the left (heavy lines), the (a) network simply reduces to the (b) network whose total resistance between P and Q is equal to $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}=\frac{11}{12}$.

Table 1. A subsequence of the calculated sequence ($2\leq b\leq 650$) of the fractal generator electric resistance R_b.

b	R_b	b	R_b
50	4.235 77	400	6.509 60
150	5.432 65	450	6.639 17
200	5.748 04	500	6.755 10
250	5.992 99	550	6.859 99
300	6.193 30	600	6.955 77
350	6.362 75	650	7.043 89

the limiting behaviour of R_b. It has been argued that resistance of a large part, of linear size b, of a triangular resistor network should be proportional to $\ln b$ (Kantor *et al* 1987). For the triangular resistor network of the fractal generator of size b a weaker statement can be straightforwardly vindicated, namely one can easily prove that R_b cannot diverge in a slower way than the function $\ln b$. To see this it is sufficient to substitute zero resistance bonds for all unit resistor bonds of the fractal generator that are leaned, for instance, to the left (see figure 1). The resulting network has resistance of the form

$$R_b^{(l)} = \left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2b}\right) \sim \tfrac{1}{2}\ln b \qquad b \to \infty \tag{5}$$

where the superscript l indicates that (5) cannot be larger than the true R_b. However, when we plot the true R_b, calculated for $2 \leqslant b \leqslant 650$, against $\ln b$ we can see that it does also diverge logarithmically (see figure 2).

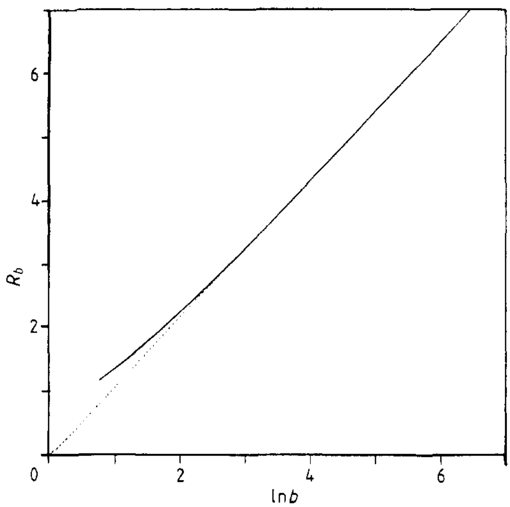

Figure 2. The exact values of R_b (full curve) plotted against $\ln b$. The slope of the dotted straight line is equal to 1.082 74, which is indeed the minimum of the ratio $R_b/\ln b$. The fact that the dotted line first lies below and then *virtually* coalesces with the full curve depicts that $R_b^{(l)}$, given by (5), is a lower bound of R_b.

In fact, from the actual table of data one can see that, for $b > 55$, R_b slightly deviates from the simple logarithmic behaviour, more specifically R_b increases marginally *faster* than a logarithmic function. This deviation, which we call $p(b) \equiv R_b - G \ln b$ (with b being a constant), can be hardly noticed from the plot of data given in figure 2. If one ignores $p(b)$ and accepts the proposition $R_b = G \ln b$ to be true for very large b, then (2b) follows straightforwardly from (3) and (4).

Suppose, on the other hand, we allow for the possibility that $p(b)$ may significantly change the form of the asymptotic law (2b) (although $p(b)$ for $b < 650$ is two orders of magnitude smaller than $G \ln b$). If we insist on the proposition $R_b = G \ln b + p(b)$, then in (2b) a leading term could appear which might be different from both $\ln(\ln b)/\ln b$ and constant$/\ln b$. The form of this possibly competing term should be $-[\ln(1 + p(b)/G \ln b)]/\ln b$. If the absolute value of this term becomes larger than

$1/\ln b$ then the plot of the function $y=(2-d_s-\ln(\ln b)/\ln b)\ln b$ against $x=1/\ln b$ would increase with decreasing x. In fact, from figure 3 we can see that, according to our data, the function y most likely tends to some constant value when $x\to 0$, i.e. when $b\to\infty$. This means that the term $-[\ln(1+p(b)/G\ln b)]/\ln b$ cannot predominate the terms already present in (2*b*). Therefore, if we do not want to consider an asymptotic region of b larger than one in which (2*b*) is correct, we can neglect the deviation $p(b)$.

To confirm (2*b*), we consider the set of the exact data for R_b that we have calculated ($2\leqslant b\leqslant 650$). To this end we take on the procedure applied previously by Borjan *et al* (1987), i.e. we group our data for larger b into successive intervals of 51 and perform independent least-square fittings, for each interval, to the formulae (2*a*) and (2*b*). However, in order to have the same number of fitting constants in both cases (in (2*a*) these are B and β), we write (2*b*) in the form

$$d_s \simeq 2-\frac{\gamma\ln(C'\ln b)}{\ln b} \tag{6}$$

where γ and C' are now the fitting constants. The latter form is equivalent to assuming that $R_b\simeq C(\ln b)^{\gamma}$ (for large b), with C' being equal to $C^{1/\gamma}$. Our analysis reveals that γ tends to one and formula (6) gives persistently better fittings, to the exact data for R_b, than formula (2*a*) (see figure 4). However, we may conclude that (6), and thereby (2*b*), should be regarded as a two-parameter approximation for the spectral dimension of the fractals with large b.

Comparing the asymptotic law (2*b*) with the asymptotic law for the fractal dimension, $d_f\simeq 2-\ln 2/\ln b$, one can speculate that the dynamical properties of the fractals under study approach their Euclidean counterpart in a slower way than the

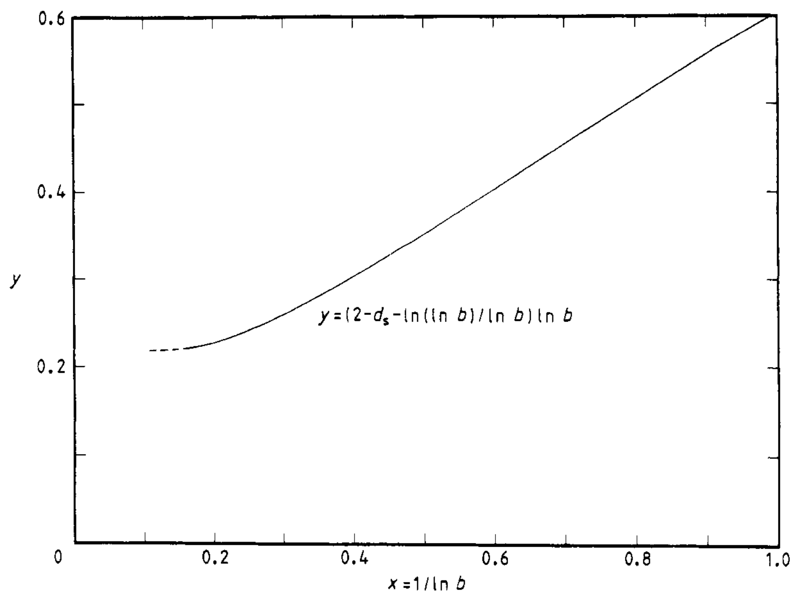

Figure 3. The plot of our exact data in the form $y=(2-d_s-\ln(\ln b)/\ln b)\ln b$ against $x=1/\ln b$. One can see that, with increasing b, the asymptotic law (2*b*) gradually becomes valid in such a way that there can hardly be an additional term that makes y increase when $x=1/\ln b$ decreases.

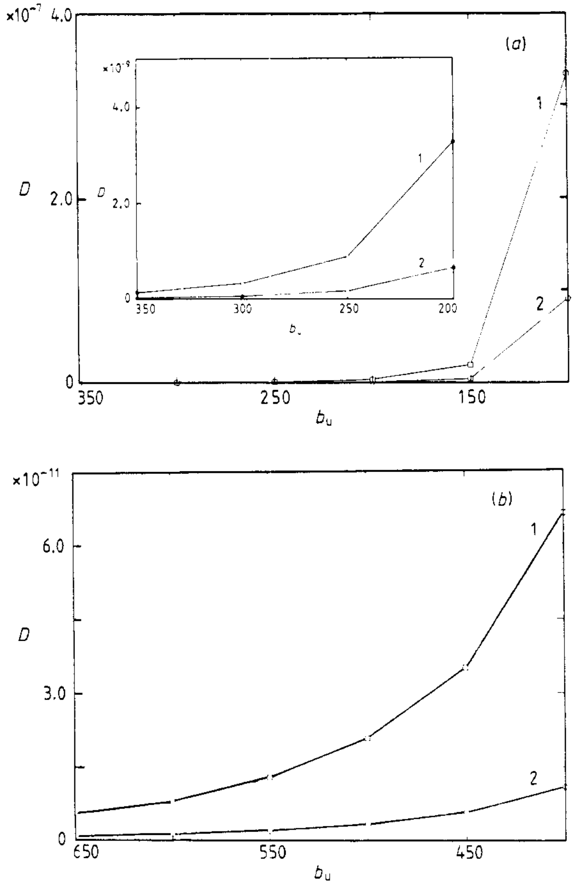

Figure 4. Plot of the mean square deivations D of the spectral dimension d_s evaluated according to equations ($2a$) and (6), with the fitting constants determined from exact values of d_s. Intervals of 51 values each have been used. The upper bounds of intervals are designated as b_u. Curve 1 corresponds to the asymptotic formula described by equation ($2a$), whereas curve 2 corresponds to the asymptotic formula described by equation (6). As b_u increases, D decreases drastically. To allow a better observation of this decrease the data have been plotted in two separate graphs.

way in which geometry of the fractals approach the Euclidean geometry. Indeed, the same asymptotic law ($2b$) has been suggested by Dhar (1987) who studied the self-avoiding walks (SAW) on the Sierpinski type of fractals. By using the finite-size scaling theory, Dhar (1987) extended the exact analysis performed for $2 \leqslant b \leqslant 8$ (Elezović *et al* 1987) to very large b and found that critical exponents of SAW on the fractals, compared with those on regular two-dimensional lattices, have the first-order corrections proportional to $\ln(\ln b)$ and the second-order corrections proportional to $1/\ln b$. In other words, the first-order corrections are proportional to the first-order deviations of the spectral dimension d_s from its Euclidean value. At present, there is no theory, or hypothesis, which claims that this type of behaviour at the fractal-lattice crossover should be universal for all critical phenomena. For this reason, it is challenging to find an example of criticality whose properties at the fractal-lattice crossover have the first-order corrections proportional to $1/\ln b$.

One of the authors (SM) would like to express his gratefulness to the Center for Polymer Studies for the cordial hospitality during his stay at the Boston University where a large portion of this work was done. We also wish to acknowledge helpful discussions with F Wu, J Nittmann, J Given, D Ben-Avraham and M Kardar. This work has been supported in part through funds provided by the Yugoslav-USA Joint Scientific Board under project no 534 (NSF) and by the Serbian Science Foundation under project no 1.2.

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